Better synchronizability predicted by crossed double cycle

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In this Brief Report, we propose a network model named crossed double cycles, which are completely symmetrical and can be considered as the extensions of nearest-neighboring lattices. The synchronizability, measured by eigenratio R, can be sharply enhanced by adjusting the only parameter, the crossed length m. The eigenratio R is shown very sensitive to the average distance L, and the smaller average distance will lead to better synchronizability. Furthermore, we find that, in a wide interval, the eigenratio R approximately obeys a power-law form as $R \sim L^{1.5}$.

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Synchronization is observed in a variety of natural, social, physical, and biological systems [1], and has found applications in a variety of fields including communications, optics, neural networks, and geophysics [2–7]. The large networks of coupled dynamical systems that exhibit synchronized state are subjects of great interest. In the early stage, the corresponding studies are restricted to either the regular networks [8,9], or the random ones [10,11]. However, recent empirical studies have demonstrated that many real-life networks cannot be treated as regular or random networks. The most important two of their common statistical characteristics are called the small-world effect [12] and scale-free property [13]. Therefore, very recently, most of the studies about network synchronization have focused on complex networks, and have found that the networks of small-world effect and scale-free property may be easier to synchronize than regular lattices [14–16].

One of the ultimate goals in studying network synchronization is to understand how the network topology affects the synchronizability. In the simplest case (see below), the network synchronizability can be measured well by the eigenratio R [17–20]; thus, the above question degenerates to understanding the relationship between network structure and its eigenvalues. Since there are countless topological characters for networks, a natural question is addressed: What is the most important factor by which the synchronizability of the system is mainly determined? Some previous works indicated the average distance L [21] is one of the key factors. However, consistent conclusions have not been achieved [19,22–25]. Another extensively studied one is network heterogeneity, which can be measured by the variance of degree distribution or betweenness distribution [26,27]. Some detailed comparisons among various networks have been done, indicating the network synchronizability will be better with smaller heterogeneity [24,28,29]. However, a well-known counterexample is the regular networks with homogeneous structure that display very poor synchronizability. Because the networks used for comparison in previous studies are of both varying average distances and degree variances, strict and clear conclusions cannot be achieved. In addition, some researchers deem that the more intrinsic ingredient leading to better synchronizability is the randomicity [30]; that is to say, the intrinsic reason making small-world and scale-free networks having better synchronizability than regular ones is their random structures. Therefore, if one wants to show clearly how L affects the network synchronizability, he should investigate the networks of different L but with the same degree variance. If he wants to assert that it is not the randomicity but smaller (or longer) L resulting in the better synchronizability, deterministic networks are required.

In this Brief Report, we proposed a deterministic network model named *crossed double cycles* (CDCs for short). The CDCs are of degree variance equal to zero, and by adjusting the only parameter m, named the crossed length, the average distance of CDCs can be changed. By using this ideal model, we demonstrate that the smaller L will result in better synchronizability, and provide a useful method to enhance the synchronizability of nearest-neighbor coupling networks.

In network language [31], the cycle C_N denotes a network consisting of N vertices $x_1,x_2,\ldots,x_N$. These N vertices are arranged as a ring, and the nearest two vertices are connected. Hence, C_N has N edges connecting the vertices $x_1x_2,x_2x_3,\ldots,x_{N-1}x_N$, and x_Nx_1. The CDCs, denoted by $G(N,m)$, can be constructed by adding two edges, called crossed edges, to each vertex in C_N. The two vertices connecting by a crossed edge are of distance m in C_N. For example, the network $G(N,3)$ can be constructed from C_N by connecting $x_1x_4,x_2x_5,\ldots,x_{N-1}x_2$, and x_Nx_3. The network $G(N,2)$ is isomorphic [32] to a one-dimensional lattice with periodic boundary conditions wherein each vertex connects to its nearest and next-nearest neighbors. A sketch map of $G(20,4)$ is shown in Fig. 1.

Clearly, all the vertices in $G(N,m)$ are of degree 4; thus, the degree variance is equal to 0. Furthermore, $G(N,m)$ is vertex transitive; that is to say, for any two vertices x and y in $G(N,m)$, there exists an automorphism mapping $\theta: V(G)\to V(G)$ such that $y=\theta(x)$. The vertex-transitivity networks are completely symmetric, which is of particular practicability in the design of topological structures of data memory allocation and multiple processor systems [33].

Denoting $L(k)$ the average distance of C_{k+1}, we have

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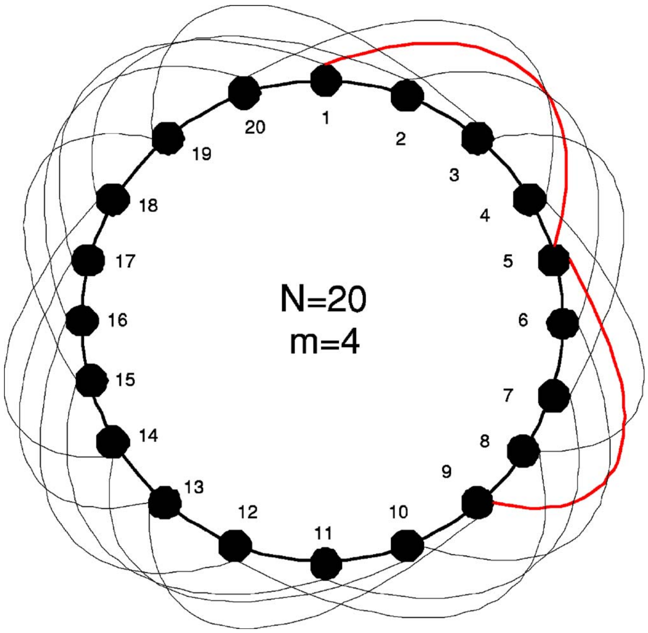

FIG. 1. (Color online) The sketch maps of $G(20,4)$.

$$L(k)=\begin{cases}\dfrac{k+2}{4}, & k \text{ is even},\\[2mm] \dfrac{(k+1)^2}{4k}, & k \text{ is odd}.\end{cases} \qquad (1)$$

For $N \gg m$, we assume N can be exactly divided by m and denote $k=\frac{N}{m}$. Since $G(N,m)$ is vertex transitive, the average distance of $G(N,m)$ is equal to the average distance between vertex x_1 to all other vertices. The network $G(N,m)$ contains k end-to-end C_{m+1} as $x_1x_2\cdots x_{m+1}, x_{m+1}x_{m+2}\cdots x_{2m+1}, \ldots, x_{N+1-m}\cdots x_Nx_1$. Going from the vertex x_1 to a certain vertex x_i can then be divided to two processes. Firstly, travel through the crossed edges to the nearest vertex that belongs to x_i's cycle mentioned above. Secondly, pass by a shortest path restricted in this cycle to x_i. For example, the path from x_1 to x_{10} in $G(20,4)$ is $x_1 \rightarrow x_5 \rightarrow x_9 \rightarrow x_{10}$. The first two edges are the crossed edges and identified by red lines in Fig. 1, and the last edge is in the cycle $x_9x_{10}x_{11}x_{12}x_{13}$. Hence, one can obtain the average distance of $G(N,m)$ for $N \gg m$ as

$$L_G(N,m)=L(m)+L(k)-1. \qquad (2)$$

Figure 2 shows the simulation results of $L_G(N,m)$ for $N=1000,2000,3000,4000$, and $m \leqslant 100$, which agree accurately with the analytic ones.

In succession, we investigate the changes of CDCs' synchronizability with m. Consider N identical dynamical systems (oscillators) with the same output function, which are located on the vertices of a network and coupled linearly and symmetrically with neighbors connected by edges of the network. The coupling fashion ensures the synchronization manifold an invariant manifold, and the dynamics can be locally linearized near the synchronous state. The state of the ith oscillator is described by $\mathbf{x}^i$, and the set of equations of

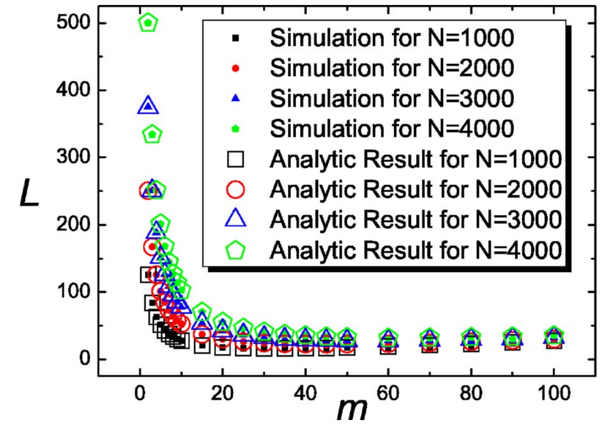

FIG. 2. (Color online) The average distance of the CDCs. The black squares, red circles, blue triangles, and green pentagons represent the cases of $N=1000$, 2000, 3000, and 4000, respectively. The smaller solid symbols and larger hollow symbols represent the simulation and analytic results, respectively.

motion governing the dynamics of the N coupled oscillators is

$$\dot{\mathbf{x}}^i=\mathbf{F}(\mathbf{x}^i)-\sigma\sum_{j=1}^{N}G_{ij}\mathbf{H}(\mathbf{x}^j), \qquad (3)$$

where $\dot{\mathbf{x}}^i=\mathbf{F}(\mathbf{x}^i)$ governs the dynamics of individual oscillator, $\mathbf{H}(\mathbf{x}^j)$ is the output function and σ is the coupling strength. The $N\times N$ Laplacian $\mathbf{G}$ is given by

$$G_{ij}=\begin{cases}k_i, & \text{for } i=j,\\ -1, & \text{for } j\in\Lambda_i,\\ 0, & \text{otherwise}.\end{cases} \qquad (4)$$

Because of the positive semidefinite of $\mathbf{G}$, all the eigenvalues of it are non-negative reals and the smallest eigenvalue θ_0 is always zero, for the rows of $\mathbf{G}$ have zero sum. Thus, the eigenvalues can be ranked as $\theta_0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_{N-1}$. The ratio of the maximum eigenvalue θ_{N-1} to the smallest nonzero one θ_1 is widely used to measure the synchronizability of the network [17,19], if the eigenratio $R=\theta_{N-1}/\theta_1$ satisfies

$$R < \alpha_2/\alpha_1, \qquad (5)$$

we say the network is synchronizable. The right-hand side of this inequality depends only on the dynamics of individual oscillator and the output function [19], while the eigenratio R depends only on the Laplacian $\mathbf{G}$. R indicates the synchronizability of the network, the smaller it is the better synchronizability and vice versa. In this Brief Report, for universality, we will not address a particular dynamical system, but concentrate on how the network topology affects eigenratio R.

Since the Laplacian for any CDC is shift invariant, the eigenvalues can be calculated from a discrete Fourier transform of a row of the Laplacian matrix [20]. Denote γ_i $(i=0,1,\ldots,N-1)$ the ith eigenvalue [34], it reads

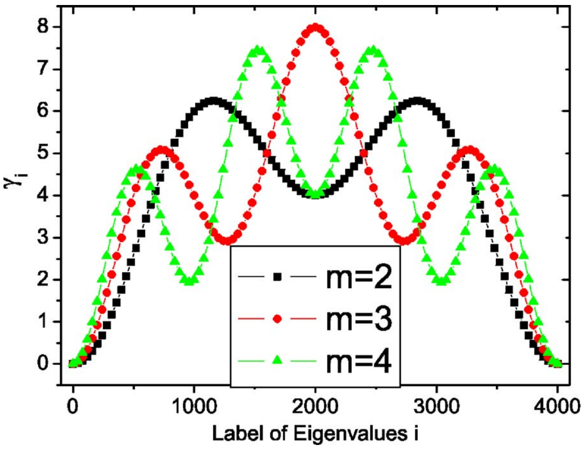

FIG. 3. (Color online) Eigenvalues. The network size $N=4000$ is fixed, and the black squares, red circles, and green triangles denote the numerical results of eigenvalues for the cases $m=2$, $m=3$, and $m=4$, respectively. The corresponding curves represent the analytical solutions.

$$\gamma_i = 2\left(2 - \cos\frac{2\pi i}{N} - \cos\frac{2\pi i m}{N}\right). \tag{6}$$

Figure 3 shows the numerical results of eigenvalues, which accurately agree with the analytical solutions. Clearly, for odd m and even N, the maximal eigenvalue is $\theta_{N-1}=\gamma_{N/2}=8$. However, if m is even or N is odd, the maximal eigenvalue θ_{N-1} is smaller than 8 and cannot be expressed sententiously. The smallest nonzero eigenvalue θ_1 equals γ_1, and can be approximately obtained under the condition $N \gg m$

$$\theta_1 = \gamma_1 \approx \frac{4\pi^2}{N^2}(1+m^2). \tag{7}$$

Figure 4 reports γ_1 as a function of m, which scales as m^2, as predicted by Eq. (7), before reaching a cutoff point m_c. The

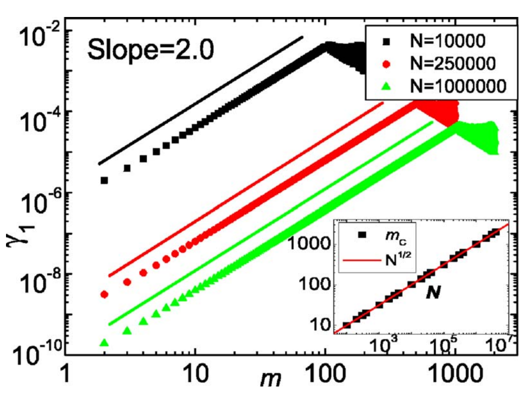

FIG. 4. (Color online) The smallest nonzero eigenvalue $\gamma_1(\theta_1)$ vs m. The black squares, red circles, and green triangles denote the cases $N=10\,000$, $N=250\,000$, and $N=1\,000\,000$, respectively. The solid lines are of slope 2 for comparison. The inset shows the cutoff point m_c as a function of network size.

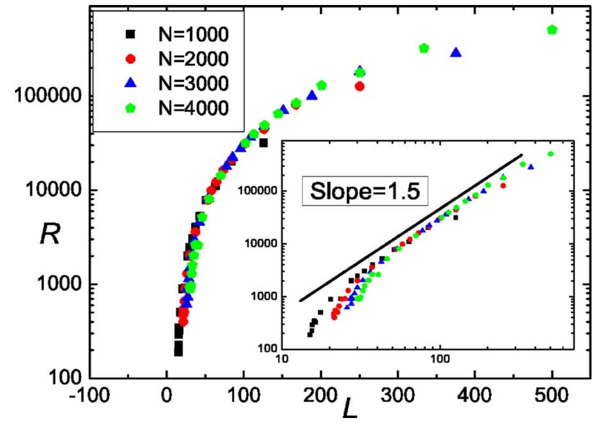

FIG. 5. (Color online) R vs L. The black squares, red circles, blue triangles, and green pentagons represent the cases of $N=1000$, 2000, 3000, and 4000, respectively. The inset shows the same data in log-log plot, indicating that the eigenratio R approximately obeys a power-law form as $R \sim L^{1.5}$. The solid line is of slope 1.5 for comparison.

numerical value of m_c is also shown in the inset of Fig. 4, which accurately obeys the form $m_c=\sqrt{N}$.

Because of the cutoff in θ_1, the fluctuations in θ_{N-1} for even m, and the relatively complex relationship between m and L, we cannot obtain a straightforward expression to comprehensively depict the relationship between R and L. In Fig. 5, we only report the numerical results about how the average distance affects the network synchronizability. One can see clearly that the network synchronizability is very sensitive to the average distance; as the increase of L, the eigenratio R sharply spans more than three magnitudes. In addition, the network synchronizability is remarkably enhanced by reducing L. When the crossed length m is not very small or very large (comparing with N), the networks with the same average distance have approximately the same synchronizability no matter what the network size is. More interesting, the calculated results indicate that the eigenratio R approximately obeys a power-law form as $R \sim L^{1.5}$ in a wide interval of L (see the inset of Fig. 5).

To sum up, we propose an ideal network model, and investigate its synchronizability. The results indicate that the average distance is an important factor affecting the network synchronizability greatly. The smaller average distance will lead to better synchronizability. This is similar to the communication systems, wherein the average distance is one of the most important parameters to measure the transmission delay (or time delay) encountered by a message traveling through the network from its source to destination, and the smaller average distance means higher efficiency for homogeneous networks. Very recently, by numerical studies, some authors think that there may exist some common features between dynamics on communication networks (traffic and diffusion) and network synchronization [29,35–37]. Since in the former dynamics shorter L will lead to greater throughput and fast spread, the underlying common features provide a possible explanation why a shorter average distance corresponds to better synchronizability.

The CDCs are natural extensions of the lattice of nearest neighbors, they are symmetric and with better synchronizability, thus have great potential in the applications for designing of topological structures of distributed processing systems, local area networks, data memory allocation and data alignment in single instruction multiple data processors [33]. In fact, the processor network of one kind of the earliest parallel processing computers is $G(16,4)$ [38]. In addition to synchronization, the cross method also has been applied in communication systems. For example, the crossed cubes have much larger throughput than hypercubes, and thus are widely used in designing parallel computing networks [39,40].

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